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Infinitesimal transformations about solutions of KdV and sine-Gordon equations through a Lie product

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Abstract. It is shown that the double infinity of infinitesimal transformations about solutions of the KdV equations which can be generated by a recursion operator and its inverse can also be obtained through a Lie product. Each of the doubly infinite hierarchies of nonlinear evolution equations obtained from these infinitesimal transformations is shown to be integrable. Similar results hold for the sine-Gordon equation.

1. Introduction

A necessary condition for the integrability of a nonlinear evolution equation (NLEE)

$$u_t(x, t) = K(u) \tag{1.1}$$

is the existence of an infinity of infinitesimal transformations (IT) about any solution $u(x, t)$ of (1.1). One way of establishing this is to find a recursion operator $T(u)$ which acting on an IT again gives an IT (Wadati 1978, Fuchssteiner and Fokas 1981 and references therein, Aiyer 1983). The recursion operator is also called a strong symmetry (Fuchssteiner 1979). An infinity of IT for the Benjamin–Ono (BO) equation and the Kadomtsev–Petviashvili (KP) equation has been obtained by an entirely novel approach (Fokas and Fuchssteiner 1981, Oevel and Fuchssteiner 1982). They have found a function $\tau(u)$ such that if $y(u)$ is an IT about $u(x, t)$ then so is the Lie product $[y(u), \tau(u)]$. $u(x, t)$ is a solution of the particular NLEE. $y(u)$ and $\tau(u)$ depend on (x, t) through $u(x, t)$, its integrals and partial derivatives with respect to x. The product $[,]$ is defined by

$$[f(u), g(u)] = (\partial/\partial\varepsilon)\{f(u+\varepsilon g) - g(u+\varepsilon f)\}_{\varepsilon=0}. \tag{1.2a}$$

$[,]$ is anticommutative and satisfies the Jacobi identity (Magri 1976),

$$[[f(u), g(u)], h(u)] + \text{cyclic terms} = 0. \tag{1.2b}$$

Since $K_0(u) \equiv u_x(x, t)$ is usually an IT about a solution $u(x, t)$ of (1.1), $K_n(u)$, $n \in N_0$ (N_0 is the set of all non-negative integers) are IT about $u(x, t)$ where

$$K_n(u) \equiv [K_{n-1}(u), \tau(u)]. \tag{1.3}$$

This approach is attractive because $y(u)$ is an IT about a solution of (1.1) if and only if

$$[y(u), K(u)] = 0. \tag{1.4}$$

If therefore

$$[K_n(u), K_m(u)] = 0, \qquad n, m \in N_0, \tag{1.5}$$

holds, then the equation $u_t = K_n(u)$ for all $n \in N_0$ has an infinity of IT. The definition (1.3) of $K_n(u)$ through a Lie product leads to a direct proof of (1.5) for the BO and KP equations.

It has also been shown (Fokas and Fuchssteiner 1981) that the BO equation does not have a recursion operator of the polynomial type as do the KdV and sine-Gordon (SG) equations. The question arises whether the generation of an infinity of IT by a recursion operator on the one hand and by a Lie product on the other are mutually exclusive.

In this paper we obtain the double infinity of IT (Aiyer 1983) for the KdV and SG equations through a Lie product. This requires that we obtain a $\tau(u)$ and essentially its inverse $\tau_0(u)$ such that the repeated Lie product of u_x and $\tau(u)$ generates one infinity of IT and of u_x and $\tau_0(u)$ another. Both for the BO and KP equations a $\tau_0(u)$ exists but

$$[u_x, \tau_0(u)] = 0 \tag{1.6}$$

and one cannot obtain the second infinity of IT using $\tau_0(u)$. We call such a $\tau_0(u)$ singular, that is there exists an IT $f(u)$ such that $[\tau_0(u), f(u)] = 0$. For the KdV and SG equations we have obtained a non-singular $\tau_0^{NS}(u)$ such that

$$[K_0(u), \tau_0^{NS}(u)] \equiv [u_x, \tau_0^{NS}(u)] = K_{-1}(u) \neq 0 \tag{1.7}$$

is an IT. Repeated Lie products of $\tau_0^{NS}(u)$ and $K_{-1}(u)$ generate the second infinity of IT $K_{-m}(u)$. There also exist a singular $\tau_0^S(u)$ and a singular and non-singular $\tau(u)$, that is a $\tau^S(u)$ and a $\tau^{NS}(u)$ for these equations. The existence of both types of functions is used to prove the main result of the paper that

$$[K_n^{NS}(u), K_m^{NS}(u)] = 0 \qquad \text{for all } n, m \in Z_0, \tag{1.8}$$

where Z_0 is the set of all integers. This implies the new result that

$$u_t = K_{-n}(u), \qquad n \in N_0, \tag{1.9}$$

are integrable, that is have a double infinity of IT. The superscript in (1.8) is to indicate that the functions $K_n^{NS}(u)$ are obtained by taking Lie products with $\tau^{NS}(u)$ for n a positive integer and with $\tau_0^{NS}(u)$ for n a negative integer.

In § 2 we prove the main result (1.8) for the KdV equation. Considerable preparation is needed for the proof. In § 2.1 we write down the expressions for $\tau_0^S(u)$, $\tau_0^{NS}(u)$, $\tau_1(u)$, $\tau^S(u)$ and $\tau^{NS}(u)$. $\tau_1(u)$ is like the 'identity' element for the Lie product. We then show that $\tau_0^S(u)$ and $\tau^S(u)$ are singular in the sense explained below (1.6). In § 2.2 we write down the singular and non-singular recursion operators and show that they connect $\tau_0(u)$ to $\tau_1(u)$ and $\tau_1(u)$ to $\tau(u)$. The existence of such an operator for any NLEE may imply that it is a recursion operator for the IT about the solutions of the NLEE but we have not been able to prove this. In § 2.3 some Lie products of the various $\tau(u)$ are written down. The proofs are direct but sometimes long. In § 2.4 the functions $K_n^S(u)$ and $K_n^{NS}(u)$ for all integer n are defined recursively. The proof of (1.8) starts in § 2.5. The important steps in the proof are presented at the beginning.

In § 3 we write down the expressions for $\tau_0^S(\phi)$, $\tau_0^{NS}(\phi)$ etc for the SG equation. No proofs are given. One follows the approach presented for the KdV equation in § 2. The proofs would be simpler for the SG equation because no auxiliary dependent function need be introduced as is necessary for the KdV equation.

2. KdV equation

2.1.

The KdV equation is

$$u_t + 6uu_x + u_{3x} = 0. \tag{2.1}$$

Define

$$\begin{aligned}
\tau^{S}(u) &\equiv x(u_{3x} + 6uu_x) + 4u_{xx} + 8u^2 + 2u_x \int^x u(x_1, t)\, dx_1, \\
\tau^{NS}(u) &\equiv x(u_{3x} + 6uu_x) + 4u_{xx} + 8u^2 + 2u_x \int_{-\infty}^x u(x_1, t)\, dx_1, \\
\tau_0^{S}(u) &\equiv \psi\psi_x \int^x (1/\psi^2)\, dx_1, \\
\tau_0^{NS}(u) &\equiv \psi\psi_x \int_{-\infty}^x (1/\psi^2)\, dx_1, \\
\tau_1(u) &\equiv xu_x + 2u.
\end{aligned} \tag{2.2}$$

$\psi(x, t)$ is related to $u(x, t)$ by

$$u(x, t) = -\psi_{xx}(x, t)/\psi(x, t). \tag{2.3}$$

$\tau^{NS}(u)$ and $\tau_0^{NS}(u)$ differ from $\tau^{S}(u)$ and $\tau_0^{S}(u)$ only in the appearance of a lower limit $-\infty$ in the integrals.

The boundary condition on $\psi(x, t)$ as $x \to -\infty$, which will be needed later, is now obtained. Considering (2.3) as a differential equation in $\psi(x, t)$ with potential $u(x, t)$ and eigenvalue zero, we get $\psi(x, t) \to$ constant as $|x| \to \infty$ if $u(x, t) \to 0$ sufficiently rapidly as $|x| \to \infty$. Without loss of generality we will assume that

$$\psi(x, t) \to 1 \qquad \text{as } x \to -\infty. \tag{2.4}$$

$\tau^{S}(u)$ is termed singular because

$$[\tau^{S}(u), \psi\psi_x] = 0. \tag{2.5a}$$

But

$$[\tau^{NS}(u), \psi\psi_x] \neq 0. \tag{2.5b}$$

It is easy to verify that $\psi\psi_x$ is an IT. Evaluation of (2.5a, b) is a little involved and some details are presented below.

$$[\tau^{NS}(u), \psi\psi_x] = [x(u_{3x} + 6uu_x) + 4u_{xx} + 8u^2 + 2u_x \int_{-\infty}^x u(x_1, t)\, dx_1, \psi\psi_x], \tag{2.6}$$

$\tau^{NS}(u)$ and $\psi\psi_x$ are small changes in $u(x, t)$. However, the second element of the product (2.6) is in terms of $\psi(x, t)$. Therefore to evaluate (2.6) one has to find the variation in $\psi(x, t)$ corresponding to a variation $\tau^{NS}(u)$ in $u(x, t)$. It is convenient to transform both the elements in (2.6) as changes in $\psi(x, t)$. Thus if $u \to u + \varepsilon y$ and correspondingly $\psi \to \psi + \varepsilon z$ we have to find $z(x, t)$ in terms of $y(x, t)$. Substituting the functions $u+\varepsilon y$ and $\psi+\varepsilon z$ for u and ψ in (2.3) and comparing terms linear in ε we get

$$y=(\psi_{xx}z/\psi^2)-(z_{xx}/\psi). \tag{2.7}$$

$z(x, t)$ is evaluated with (i) $y(x, t)=\tau^{\mathrm{NS}}(u)$ and (ii) $y(x, t)=\psi\psi_x$.

(i) $\tau^{\mathrm{NS}}(u)$ is expressed in terms of $\psi(x, t)$ using (2.3). This expression is substituted for $y(x, t)$ in (2.7) and solved for $z(x, t)$. One gets

$$\tau^{\mathrm{NS}}(\psi)\equiv z=x\{\psi_{3x}-3(\psi_x\psi_{xx}/\psi)\}+2\psi_{xx}-2\psi_x\int_{-\infty}^{x}(\psi_{x_1x_1}/\psi)\,\mathrm{d}x_1. \tag{2.8}$$

It should be stressed that $\tau^{\mathrm{NS}}(\psi)$ is the variation in $\psi(x, t)$ corresponding to the variation $\tau^{\mathrm{NS}}(u)$ in $u(x, t)$. It is not $\tau^{\mathrm{NS}}(u)$ expressed in terms of $\psi(x, t)$.

(ii) With $y=\psi\psi_x$,

$$z(x, t)=\tfrac{1}{4}\psi\int_{-\infty}^{x}\{(1/\psi^2)-\psi^2\}\,\mathrm{d}x_1. \tag{2.9}$$

One can now evaluate (2.6) and after considerable algebra one gets

$$[\tau^{\mathrm{NS}}(\psi), \tfrac{1}{4}\psi\int_{-\infty}^{x}\{(1/\psi^2)-\psi^2\}\,\mathrm{d}x_1]=-\psi_x. \tag{2.10}$$

ψ_x is a variation in $\psi(x, t)$. The corresponding variation in $u(x, t)$ is u_x and can be obtained directly from (2.7), so that

$$[\tau^{\mathrm{NS}}(u), \psi\psi_x]=u_x. \tag{2.11a}$$

Similar calculation gives

$$[\tau^{\mathrm{S}}(u), \psi\psi_x]=0, \qquad [\tau_0^{\mathrm{S}}(u), u_x]=0, \tag{2.11b, c}$$

$$[\tau_0^{\mathrm{NS}}(u), u_x]=-\psi\psi_x. \tag{2.11d}$$

Variations $\tau^{\mathrm{S}}(\psi)$, $\tau_0^{\mathrm{S}}(\psi)$, $\tau_0^{\mathrm{NS}}(\psi)$ in $\psi(x, t)$ corresponding to variations $\tau^{\mathrm{S}}(u)$, $\tau_0^{\mathrm{S}}(u)$, $\tau_0^{\mathrm{NS}}(u)$ in $u(x, t)$ are given below:

$$\tau^{\mathrm{S}}(\psi)=x\left\{\psi_{3x}-3(\psi_x\psi_{xx}/\psi)\right\}+2\psi_{xx}-2\psi_x\int^{x}(\psi_{x_1x_1}/\psi)\,\mathrm{d}x_1, \tag{2.12a}$$

$$\tau_0^{\mathrm{S}}(\psi)=\tfrac{1}{4}\psi\left\{\int^{x}\mathrm{d}x_1(1/\psi^2)\int^{x_1}\psi^2\,\mathrm{d}x_2-\int^{x}\mathrm{d}x_1\,\psi^2\int^{x_1}(1/\psi^2)\,\mathrm{d}x_2\right\}. \tag{2.12b}$$

$\tau_0^{\mathrm{NS}}(\psi)$ is (2.12b) with $-\infty$ as the lower limit in the integrals.

2.2.

The singular and non-singular recursion operators are defined by (Aiyer 1983, unpublished):

$$T^{\mathrm{S}}(u)=\partial^2/\partial x^2+4u+2u_x\int^{x}\mathrm{d}x_1,$$

$$T^{\mathrm{NS}}(u)=\partial^2/\partial x^2+4u+2u_x\int_{-\infty}^{x}\mathrm{d}x_1. \tag{2.13}$$

It is easy to verify that

$$T^{\mathrm{S}}(u)\{\tau_0^{\mathrm{S}}(u)\}=T^{\mathrm{S}}(u)\{\tau_0^{\mathrm{NS}}(u)\}=-\tau_1(u), \tag{2.14a}$$

$$T^{\mathrm{S}}(u)\{\tau_1(u)\} = \tau^{\mathrm{S}}(u), \qquad T^{\mathrm{NS}}(u)\{\tau_1(u)\} = \tau^{\mathrm{NS}}(u). \tag{2.14b, c}$$

One cannot replace $T^{\mathrm{S}}(u)$ in (2.14a) by $T^{\mathrm{NS}}(u)$ as a divergent term $\int_{-\infty}^{x} 1 \cdot \mathrm{d}x_1$ then appears.

2.3.

By the method used in evaluating (2.6) or directly one has

$$[\tau^{\mathrm{S}}(u), \tau_1(u)] = 2\tau^{\mathrm{S}}(u), \qquad [\tau^{\mathrm{NS}}(u), \tau_1(u)] = 2\tau^{\mathrm{NS}}(u), \tag{2.15a, b}$$

$$[\tau^{\mathrm{S}}(u), \tau_0^{\mathrm{S}}(u)] = [\tau^{\mathrm{S}}(u), \tau_0^{\mathrm{NS}}(u)] = -4\tau_1(u). \tag{2.15c}$$

To prove (2.14c) and (2.15b) it is assumed that $u(x, t) \to 0$ faster than $x \to -\infty$ in the limit $x \to -\infty$.

2.4.

Define

$$K_0^{\mathrm{NS}}(u) = K_0^{\mathrm{S}}(u) = u_x, \tag{2.16a}$$

$$K_n^{\mathrm{S}}(u) \equiv [K_{n-1}^{\mathrm{S}}(u), \tau^{\mathrm{S}}(u)], \qquad K_n^{\mathrm{NS}}(u) \equiv [K_{n-1}^{\mathrm{NS}}(u), \tau^{\mathrm{NS}}(u)] \tag{2.16b, c}$$

and

$$K_{-1}^{\mathrm{S}}(u) = K_{-1}^{\mathrm{NS}}(u) = \psi\psi_x, \tag{2.17a}$$

$$K_{-n-1}^{\mathrm{S}}(u) \equiv [K_{-n}^{\mathrm{S}}(u), \tau_0^{\mathrm{S}}(u)], \qquad K_{-n-1}^{\mathrm{NS}}(u) \equiv [K_{-n}^{\mathrm{NS}}(u), \tau_0^{\mathrm{NS}}(u)]. \tag{2.17b, c}$$

A few words about the choice of $K_{-1}^{\mathrm{S}}(\psi)$ and $K_{-1}^{\mathrm{NS}}(\psi)$, the variation in $\psi(x, t)$ corresponding to a variation $K_{-1}^{\mathrm{S}}(u) = K_{-1}^{\mathrm{NS}}(u)$ in $u(x, t)$. Using (2.7), $K_{-1}(\psi)$ can have three forms:

(i) $\quad -\frac{1}{4}\psi \int^{x} \psi^2 \, \mathrm{d}x_1,$

(ii) $\quad \frac{1}{4}\psi \int^{x} \{(1/\psi^2) - \psi^2\} \, \mathrm{d}x_1,$

(iii) $\quad \frac{1}{4}\psi \int_{-\infty}^{x} \{(1/\psi^2) - \psi^2\} \, \mathrm{d}x_1.$

For all these $K_{-1}(\psi)$,

$$[K_{-1}(\psi), \tau^{\mathrm{S}}(\psi)] = 0, \qquad [K_{-1}(\psi), \tau^{\mathrm{NS}}(\psi)] = \psi_x. \tag{2.18a, b}$$

To obtain $K_{-n}^{\mathrm{NS}}(u)$ and $K_{-n}^{\mathrm{S}}(u)$ from (2.17b, c), $K_{-n}^{\mathrm{NS}}(\psi)$ and $K_{-n}^{\mathrm{S}}(\psi)$ have to be evaluated first. This requires that $K_{-1}^{\mathrm{NS}}(\psi)$ and $K_{-1}^{\mathrm{S}}(\psi)$ be first defined and here we have a choice. We choose

$$K_{-1}^{\mathrm{S}}(\psi) = \tfrac{1}{4}\psi \int^{x} \{(1/\psi^2) - \psi^2\} \, \mathrm{d}x_1, \tag{2.19a}$$

$$K_{-1}^{\mathrm{NS}}(\psi) = \tfrac{1}{4}\psi \int_{-\infty}^{x} \{(1/\psi^2) - \psi^2\} \, \mathrm{d}x_1. \tag{2.19b}$$

This choice of $K^{\mathrm{S}}_{-1}(\psi)$ simplifies to some extent the proof of (1.8). The choice of $K^{\mathrm{NS}}_{-1}(\psi)$ is made on the basis that

$$[K_0(\psi), \tau_0^{\mathrm{NS}}(\psi)] = \tfrac{1}{4}\psi \int_{-\infty}^{x} \{(1/\psi^2) - \psi^2\}\, \mathrm{d}x_1 = K^{\mathrm{NS}}_{-1}(\psi), \tag{2.20}$$

where $K_0(\psi) = \psi_x$ is the variation in $\psi(x, t)$ corresponding to a variation $K_0(u) = u_x$ in $u(x, t)$. Thus we connect $K^{\mathrm{NS}}_n(\psi)$ for positive and negative n.

2.5.

The main steps in the proof are to show the following.

(a) $[K^{\mathrm{S}}_{-n}(u), u_x] = 0$, for all $n \in N_0$.

(b) $[K^{\mathrm{NS}}_{-n}(u), u_x] = 0$, $n \in N_0$, using (a). This is the longest part of the proof.

(c) $[K^{\mathrm{NS}}_{-n}(u), K^{\mathrm{NS}}_{-m}(u)] = 0$, $m, n \in N_0$, by taking Lie products of (b) with $\tau_0^{\mathrm{NS}}(u)$.

(d) $[K^{\mathrm{S}}_n(u), \psi\psi_x] = 0$, $n \in N_0$.

(e) $K^{\mathrm{S}}_n(u) = K^{\mathrm{NS}}_n(u)$, $n \in N_0$.

(f) $[K^{\mathrm{NS}}_n(u), K^{\mathrm{NS}}_m(u)] = 0$, $m, n \in N_0$, using (d), (e) and taking Lie products with $\tau^{\mathrm{NS}}(u)$.

(g) $[K^{\mathrm{NS}}_n(u), K^{\mathrm{NS}}_{-m}(u)] = 0$, $n, m \in N_0$, using (2.15*c*) and by taking the Lie product of $[K^{\mathrm{NS}}_n(u), K^{\mathrm{NS}}_{-1}(u)]$ with $\tau_0^{\mathrm{NS}}(u)$.

Combining (c), (f) and (g) completes the proof of (1.8).

Proof of (a). Assume that

$$[K^{\mathrm{S}}_{-n+1}(u), u_x] = 0 \qquad \text{for some } n \geqslant 2.$$

Taking the Lie product with $\tau_0^{\mathrm{S}}(u)$ and using the Jacobi identity (1.2*b*) we have

$$0 = [[K^{\mathrm{S}}_{-n+1}(u), u_x], \tau_0^{\mathrm{S}}(u)] = -[[\tau_0^{\mathrm{S}}(u), K^{\mathrm{S}}_{-n+1}(u)], u_x] - [[u_x, \tau_0^{\mathrm{S}}(u)], K^{\mathrm{S}}_{-n+1}(u)].$$

Using (2.11*c*) and (2.17*b*) one has

$$[K^{\mathrm{S}}_{-n}(u), u_x] = 0. \tag{2.21}$$

That $[K_{-1}(u), u_x] = 0$ can be easily verified and (2.21) follows by induction.

Proof of (b). It is first shown that $K^{\mathrm{S}}_{-n-1}(\psi)$ can be written as an operator acting on $K^{\mathrm{S}}_{-n}(\psi)$ for $n \geqslant 1$ where, $K^{\mathrm{S}}_{-n}(\psi)$ is the variation in $\psi(x, t)$ due to a variation $K^{\mathrm{S}}_{-n}(u)$ in $u(x, t)$. From (2.12*b*) and (2.17*b*) it follows that

$$\begin{aligned} K^{\mathrm{S}}_{-n-1}(\psi) &= [K^{\mathrm{S}}_{-n}(\psi), \tau_0^{\mathrm{S}}(\psi)] \\ &= \left[K^{\mathrm{S}}_{-n}(\psi), \psi \int^{x} \mathrm{d}x_1 (1/\psi^2) \int^{x_1} \psi^2\, \mathrm{d}x_2 - \psi \int^{x} \mathrm{d}x_1\, \psi^2 \int^{x_1} (1/\psi^2)\, \mathrm{d}x_2 \right]. \end{aligned} \tag{2.22}$$

It will be shown that

$$K^{\mathrm{S}}_{-n-1}(\psi) = \tfrac{1}{4} C_{-n} \psi \left\{ \int^{x} \mathrm{d}x_1 (1/\psi^2) \int^{x_1} \psi^2 Z^{\mathrm{S}}_{-n}(\psi)\, \mathrm{d}x_2 + \int^{x} \mathrm{d}x_1\, \psi^2 \int^{x_1} (Z^{\mathrm{S}}_{-n}(\psi)/\psi^2)\, \mathrm{d}x_2 \right\}, \tag{2.23}$$

where $C_{-n} = C_{-n+1} + 4$ and

$$\psi Z^{\mathrm{S}}_{-n}(\psi) = K^{\mathrm{S}}_{-n}(\psi). \tag{2.24}$$

Expanding the Lie product (2.22) and using (2.23) one has to show that

$$(\partial/\partial\varepsilon)\,\{Z^{S}_{-n}(\psi+\varepsilon\tau^{S}_{0}(\psi))\}_{\varepsilon=0}+2\int^{x}(Z^{S}_{-n}/\psi^{2})\,dx_{1}\int^{x}\psi^{2}\,dx_{2}$$

$$+2\int^{x}\psi^{2}Z^{S}_{-n}\,dx_{1}\int^{x}(1/\psi^{2})\,dx_{2}$$

$$=(C_{-n}+4)\left\{\int^{x}dx_{1}(1/\psi^{2})\int^{x_{1}}\psi^{2}\,Z^{S}_{-n}\,dx_{2}+\int^{x}dx_{1}\,\psi^{2}\int^{x_{1}}(Z^{S}_{-n}/\psi^{2})\,dx_{2}\right\}. \tag{2.25}$$

The proof is by induction. Assume (2.23) holds for $n=n+1$ that is

$$\psi Z^{S}_{-n}(\psi)=K^{S}_{-n}(\psi)$$

$$=[K^{S}_{-n+1}(\psi),\ \tau^{S}_{0}(\psi)]$$

$$=\tfrac{1}{4}C_{-n+1}\psi\left\{\int^{x}dx_{1}(1/\psi^{2})\int^{x_{1}}\psi^{2}Z^{S}_{-n+1}\,dx_{2}\right.$$

$$\left.+\int^{x}dx_{1}\,\psi^{2}\int^{x_{1}}(Z^{S}_{-n+1}/\psi^{2})\,dx_{2}\right\}. \tag{2.26}$$

Then (2.25) is true with $n=n-1$, that is,

$$(\partial/\partial\varepsilon)\{Z_{-n+1}(\psi+\varepsilon\tau^{S}_{0}(\psi))\}_{\varepsilon=0}+2\int^{x}(Z^{S}_{-n+1}/\psi^{2})\,dx_{1}\int^{x}\psi^{2}\,dx_{2}$$

$$+2\int^{x}\psi^{2}Z^{S}_{-n+1}\,dx_{1}\int^{x}(1/\psi^{2})\,dx_{2}$$

$$=(C_{-n+1}+4)\left\{\int^{x}dx_{1}\,\psi^{2}\int^{x_{1}}(Z^{S}_{-n+1}/\psi^{2})\,dx_{2}\right.$$

$$\left.+\int^{x}dx_{1}(1/\psi^{2})\int^{x_{1}}\psi^{2}Z^{S}_{-n+1}\,dx_{2}\right\}. \tag{2.27}$$

Use (2.26) to write $Z^{S}_{-n}(\psi)$ in terms of $Z^{S}_{-n+1}(\psi)$ in (2.25). Eliminate $(\partial/\partial\varepsilon)\{Z^{S}_{-n+1}(\psi+\varepsilon\tau^{S}_{0}(\psi))\}_{\varepsilon=0}$ using (2.27) and assume that $(\partial/\partial\varepsilon)(\quad)|_{\varepsilon=0}$ and the integral $\int^{x}dx_{1}$ commute. After some algebra one proves (2.25). The inductive proof is completed by showing that

$$\psi Z^{S}_{-2}(\psi)=K^{S}_{-2}(\psi)$$

$$=[K^{S}_{-1}(\psi),\ \tau^{S}_{0}(\psi)]$$

$$=\tfrac{1}{2}\psi\left\{\int^{x}dx_{1}(1/\psi^{2})\int^{x_{1}}\psi^{2}Z^{S}_{-1}\,dx_{2}+\int^{x}dx_{1}\,\psi^{2}\int^{x_{1}}(Z^{S}_{-1}/\psi^{2})\,dx_{2}\right\}, \tag{2.28}$$

where

$$\psi Z^{S}_{-1}(\psi)=K^{S}_{-1}(\psi)=\tfrac{1}{4}\psi\int^{x}\{(1/\psi^{2})-\psi^{2}\}\,dx_{1}$$

from (2.19a). $C_{-1}=2$ and C_{n} is determined. The proof of (2.28) is direct but long.

It can similarly be shown that

$$K^{NS}_{-n-1}(\psi)=[K^{NS}_{-n}(\psi),\ \tau_0^{NS}(\psi)]$$
$$=\tfrac{1}{4}C_{-n}\psi\left\{\int_{-\infty}^{x}\mathrm{d}x_1(1/\psi^2)\int_{-\infty}^{x_1}\psi^2 Z^{NS}_{-n}(\psi)\,\mathrm{d}x_2\right.$$
$$\left.+\int_{-\infty}^{x}\mathrm{d}x_1\,\psi^2\int_{-\infty}^{x_1}(Z^{NS}_{-n}(\psi)/\psi^2)\,\mathrm{d}x_2\right\}, \qquad (2.29)$$

where

$$\psi Z^{NS}_{-n}(\psi)=K^{NS}_{-n}(\psi).$$

We are now ready to show that

$$[K^{NS}_{-n}(\psi),\ \psi_x]=0 \qquad \text{for all } n\in N_0. \qquad (2.30)$$

From (2.23), (2.29) and the expressions (2.19) for $K^{S}_{-1}(\psi)$ and $K^{NS}_{-1}(\psi)$ it is seen that $K^{S}_{-n}(\psi)$ and $K^{NS}_{-n}(\psi)$ have the same terms except that the lower limit $-\infty$ occurs in the integrals in $K^{NS}_{-n}(\psi)$. Therefore $[K^{NS}_{-n}(\psi),\ \psi_x]$ will have all the terms of $[K^{S}_{-n}(\psi),\ \psi_x]$ (except that the integrals will have a lower limit $-\infty$) together with any terms arising from the contribution of the lower limit. Such a contribution will come only from the last integral, that is, in evaluating the change in $\int_{-\infty}^{x}\{(1/\psi^2)-\psi^2\}\,\mathrm{d}x_1$ when $\psi\to\psi+\varepsilon\psi_x$. This contribution is $-2\int\{\psi\psi_x+(\psi_x/\psi_3)\}\,\mathrm{d}x_1$ evaluated at $-\infty$. With the boundary condition (2.4) this value is zero. This proves (2.30).

Reverting to variations in $u(x,t)$, (2.30) gives

$$[K^{NS}_{-n}(u),\ u_x]=0 \qquad \text{for all } n\in N_0. \qquad (2.31)$$

This proves (b).

Proof of (*c*). Assume that

$$[K^{NS}_{-n}(u),\ K^{NS}_{-m}(u)]=0 \qquad (2.32)$$

for all $n\geqslant 1$ and for some positive integer m. Taking the Lie product with $\tau_0^{NS}(u)$, using the Jacobi identity (1.2*b*), (2.17*c*) and (2.32) it follows that

$$[K^{NS}_{-n}(u),\ K^{NS}_{-m-1}(u)]=0, \qquad n\geqslant 1. \qquad (2.33)$$

The inductive proof is complete using (2.20) and (2.31).

Proof of (*d*). The proof is identical to that of (a). Start with

$$[K^{S}_{n-1}(u),\ \psi\psi_x]=0 \qquad \text{for some } n\geqslant 2. \qquad (2.34)$$

Take the Lie product with $\tau^{S}(u)$, use the Jacobi identity (1.2*b*), (2.11*b*) and (2.16*b*) to prove

$$[K^{S}_{n}(u),\ \psi\psi_x]=0. \qquad (2.35)$$

Proof of (*e*). Following the method used to prove (2.23) it can be shown that

$$K^{S}_{n}(u)=[K^{S}_{n-1}(u),\ \tau^{S}(u)]=C_nT^{S}(u)\{K^{S}_{n-1}(u)\}$$

and

$$K^{NS}_{n}(u)=[K^{NS}_{n-1}(u),\ \tau^{NS}(u)]=C_nT^{NS}(u)\{K^{NS}_{n-1}(u)\}, \qquad (2.36)$$

where $T^{S}(u)$ and $T^{NS}(u)$ are given by (2.13) and $C_n=C_{n-1}+2$, $C_0=2$.

It has been proved (Aiyer 1983, unpublished) that if for some $n \in N_0$

$$K_n^{\mathrm{S}}(u) = K_n^{\mathrm{NS}}(u) = K_n(u)$$

then

$$K_{n+1}^{\mathrm{S}}(u) = C_{n+1} T^{\mathrm{S}}(u)\{K_n(u)\}$$
$$= C_{n+1} T^{\mathrm{NS}}(u)\{K_n(u)\} = K_{n+1}^{\mathrm{NS}}(u). \tag{2.37}$$

This proves (e) if one uses (2.16a).

Proof of (f). This is simply obtained by taking repeated Lie product of $\tau^{\mathrm{NS}}(u)$ with (2.35) and using (2.37). This result has been proved earlier by a different method (Fuchssteiner 1979).

Proof of (g). If

$$[K_n(u), \tau_1(u)] = b_n K_n(u), \tag{2.38}$$

then using the Jacobi identity, (2.15a), (2.16b) and (e),

$$[K_{n+1}(u), \tau_1(u)] = [[K_n(u), \tau^{\mathrm{S}}(u)], \tau_1(u)] = b_{n+1} K_{n+1}(u), \tag{2.39}$$

where $b_{n+1} = b_n + 2$. The inductive proof

$$[K_{n+1}(u), \tau_1(u)] = (2n+1) K_{n+1}(u)$$

is complete by noting that

$$[K_0(u), \tau_1(u)] = [u_x, \tau_1(u)] = u_x. \tag{2.40}$$

Using (2.20) and assuming that

$$[K_n(u), \tau_0^{\mathrm{NS}}(u)] = a_n K_{n-1}(u)$$

one can show that

$$[K_{n+1}(u), \tau_0^{\mathrm{NS}}(u)] = [[K_n(u), \tau^{\mathrm{S}}(u)], \tau_0^{\mathrm{NS}}(u)]$$
$$= a_{n+1} K_n(u), \tag{2.41}$$

where $a_{n+1} = a_n - 4(2n+1)$ and $a_0 = 1$.

With these results proof of (g) is simple.

Let

$$[K_n(u), K_{-m}^{\mathrm{NS}}(u)] = 0 \tag{2.42}$$

for all $n \in N_0$ and for some positive integer m; then

$$0 = [[K_n(u), K_{-m}^{\mathrm{NS}}(u)], \tau_0^{\mathrm{NS}}(u)]$$
$$= -[[\tau_0^{\mathrm{NS}}(u), K_n(u)], K_{-m}^{\mathrm{NS}}(u)] - [[K_{-m}^{\mathrm{NS}}(u), \tau_0^{\mathrm{NS}}(u)], K_n(u)].$$

The first term on the right is zero, as follows from (2.41), (2.42) and (c). Therefore using the definition (2.17c),

$$[K_n(u), K_{-m-1}^{\mathrm{NS}}(u)] = 0 \qquad \text{for all } n \in N_0. \tag{2.43}$$

The proof is completed by noting that (d) holds, that is (2.42) is true with $m = 1$.

Combining (c), (f) and (g) we have

$$[K_n^{NS}(u), K_m^{NS}(u)] = 0,$$

for all $n, m \in Z_0$. The superscripts are retained as $K_n^{NS}(u) \neq K_n^{S}(u)$ for negative n.

3. Sine-Gordon equation

For the SG equation

$$\phi_t = \int_{-\infty}^{x} \sin \phi(x_1, t)\, dx_1,$$

$$\tau_S(\phi) = x(\phi_{3x} + \tfrac{1}{2}\phi_x^3) + 2\phi_{xx} + \tfrac{1}{2}\phi_x \int^x \phi_{x_1}^2\, dx_1.$$

$\tau^{NS}(\phi)$ is the same as $\tau^S(\phi)$ with a lower limit $-\infty$ in the integral.

$$\tau_0^S(\phi) = \int^x dx_1 \exp(-i\phi(x_1, t)) \int^{x_1} \exp(i\phi(x_2, t))\, dx_2 - \text{complex conjugate}.$$

$\tau_0^{NS}(\phi)$ is $\tau_0^S(\phi)$ with $-\infty$ as the lower limit in the integrals.

The singular and non-singular recursion operators are

$$T^S(\phi) = \partial^2/\partial x^2 + \phi_x \int^x dx_1\, \phi_{x_1}\, \partial/\partial x_1$$

and $T^{NS}(\phi)$ is $T^S(\phi)$ with $-\infty$ as the lower limit.

4. Conclusion

We have shown that the double infinity of IT about any solution of the KdV equation which can be generated by a recursion operator and its inverse can also be generated by a Lie product. Using this method we are able to show that each of the hierarchy of NLEE $u_t = K_n(u)$, where $K_n(u)$ is obtained by taking repeated Lie products of u_x and $\tau^{NS}(u)$ or u_x and $\tau_0^{NS}(u)$, has a double infinity of IT. The existence of a recursion operator for IT for any one of this hierarchy of equations is not sufficient to ensure that every member of the hierarchy has an infinity of IT.

The proof depends on two factors; (1) there exist singular and non-singular $\tau(u)$ and $\tau_0(u)$, that is $\tau^S(u)$, $\tau^{NS}(u)$, $\tau_0^S(u)$ and $\tau_0^{NS}(u)$; (2) the Lie product $[K_n(u), \tau^S(u)]$ etc can be written as an operator acting on $K_n(u)$. Also the recursion operators connect $\tau_0(u)$ to $\tau_1(u)$ and $\tau_1(u)$ to $\tau(u)$.

We have proved (1.8) by a method different from that used by Fokas and Fuchssteiner (1981). This makes the proof a little long. The reason for adopting a different method mainly arises from the fact that we cannot use the idea of 'order of a function' for $K_{-n}(u)$ as these functions are not polynomials in $u(x, t)$ and its partial derivatives.

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